

Probability of Incipient Spanning Clusters in Critical Square Bond Percolation

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The probability of simultaneous occurrence of at least k spanning clusters has been studied by Monte Carlo simulations on the 2D square lattice at the bond percolation threshold $p_c = 1/2$. It is found that the probability of k and more Incipient Spanning Clusters (ISC) has the values $P(k > 1) \approx 0.00658(3)$ and $P(k > 2) \approx 0.00000148(21)$ provided that the limit of these probabilities for infinite lattices exists. The probability $P(k > 3)$ of more than three ISC could be estimated to be of the order of 10^{-11} and is beyond the possibility to compute a such value by nowadays computers. So, it is impossible to check in simulations the Aizenman law for the probabilities when $k \gg 1$. We have detected a single sample with 4 ISC in a total number of about 10^{10} samples investigated. The probability of single event is $1/10$ for that number of samples.

1 Introduction

It was a common belief until a very recent time that on 2D lattices at percolation threshold p_c there exists exactly one percolation cluster [1, 2]. Indeed, it was rigorously proven by Newman and Shulman [3] that the number of Infinite Clusters is either 0, 1 or ∞ and later it was proven [4, 5] that the infinite cluster is unique.

New insight developed recently by Aizenman, who proved [7] (see also, his talk at StatPhys 19 [6]) that the number of Incipient Spanning Clusters (ISC) in 2D critical percolation can be larger than one, and that the probability of at least k separate clusters is bounded

$$P_L(k) \begin{cases} \geq \\ \leq \end{cases} \begin{matrix} A e^{-\alpha k^2} \\ e^{-\alpha' k^2} \end{matrix} \quad (1)$$

where α and α' are positive and L is a linear lattice size. Moreover, he conjectured the existence of the limit

$$\lim_{k \rightarrow \infty} \lim_{L \rightarrow \infty} \frac{1}{k^2} \log P_L(k) = \alpha_{asympt} . \quad (2)$$

Indications of the existence of simultaneous clusters in two dimensional critical percolation in the limit of infinite lattices were found in computer simulations of Parongama Sen [8, 9] for site percolation on square lattices¹ with helical boundary conditions and in a strip geometry by Hu and Lin [10]. The detailed history of the recent development can be found in recent Stauffer mini-review [11].

Here we report on our attempt at a direct verification of Aizenman rigorous result by computer simulations.

We investigate by Monte-Carlo the number of spanning clusters in the critical bond percolation model on two dimensional square lattices. We have determined the first limit in (2), i.e. the numerical values of the probabilities

$$P(k) = \lim_{L \rightarrow \infty} P_L(k) \quad (3)$$

for $k = 1, 2$ and 3 , as shown in Fig. 3.

Using calculated probabilities we can conclude that the probability $P(k = 4)$ of four ISC is of the order of 10^{-11} and is beyond the power of today's computers. So, the numerical check of Aizenman result is a good task for the computers of XXI century.

In the Section 2 we define precisely the model and the algorithm used in simulations. The influence of subtle details of algorithms on the values of spanning probabilities was emphasized recently by Aharony and Stauffer [12]. In Section 3 we present the details of simulations. The discussion of the results is in the last section.

2 Model and Algorithm

We use in simulations a rectangular square lattice with linear size L , and exactly L sites and L bonds both horizontally and vertically. We use free

¹Actually, P. Sen found more than one spanning cluster not only in dimension 2 but also in dimensions 3,4 and 5.

boundary conditions. For clarity the example of lattice is shown in Fig. 1 together with the dual lattice. The dual lattice has the same number of sites and bonds as the original lattice (compare with [13] and [2]). This gives the possibility to keep in the finite lattices some properties of infinite lattice. First of all, the number of bonds is exactly twice the number of sites. Second, the self-duality is valid for any finite lattice size. Third, the horizontal and vertical directions are equivalent.

Each bond could be occupied with probability $q = 1 - p$ and closed with the probability p . Given the realization to each of $2 \cdot L^2$ bonds to be occupied formed the sample. Each sample could be decomposed in clusters of connected occupied bonds. For that we use Hoshen-Kopelman [14] algorithm, which is exact. We are interested in the spanning properties of such a clusters. Namely, what is the probability that a cluster connects the opposite borders of square and what is the number of disjoint spanning clusters?

An event h that the cluster spans the lattice horizontally is an event that at least one of the left sites and at least one of the right bonds are in the same cluster. The probability of such event is just the Langlands et al [13] horizontal crossing probability π_h .

For our purposes we need more detailed events. Namely, by h_1 we will denote the event that there are exactly one cluster connecting left sites with the right occupied bonds. In the same manner by h_k we will denote the event that there are exactly k disjoint clusters connecting left and right borders of our lattice. Then, the horizontal crossing probability π_h is given by $\sum_{i=1}^{\infty} \pi_{h_i}$, where we denote by π_{h_k} the probability of event h_k .

In a full analogy, by π_{v_k} we will denote the crossing probabilities from top to bottom, i.e. vertically. Obviously, for our choice of the lattice, $\pi_{h_k} = \pi_{v_k}$ for any lattice size. This could be used as a check of the calculated statistical errors.

Knowing the origin of the bad properties of both main fast methods for random number generation [15, 16] we check the results using the same linear congruential method as Langlands, et al use in [13] ($x_{i+1} = (a x_i + c) \bmod m$, with $a = 142412240584757$, $c = 11$, $m = 2^{48}$) and the shift register ($x_n = x_{n-157} \oplus x_{n-314} \oplus x_{n-471} \oplus x_{n-9689}$) [17] (the one used by Ziff in [18]) and found that the results coincide, which suggests the absence of systematic errors.

It should be noted that the variance of probabilities is independent from the lattice size because probabilities are calculated as the expectation values of corresponding indicator functions. So, we should keep the number of samples independent of the lattice size and, therefore, the only parameter

which controls the statistical errors is the number of samples M . Probabilities were calculated averaging $M = 10^8$ samples and the error bars were defined from 100 bins, each bin being the average over 10^6 samples.

Throughout the next section we deal with the critical bond percolation on square lattice ($p = p_c = 1/2$) with free boundaries.

The simulations was done under the Topos environment [21] working on a number of Digital Alpha workstations and servers in Landau Institute and Chernogolovka Science Park.

3 Numerical results

We calculate the probabilities π_{h_k} and π_{v_k} for square lattices with linear sizes $L = 8, 12, 16, 20, 30, 32, 64$ looking for all events up to 8.² Within the statistical errors the vertical and horizontal crossing probabilities coincide giving additional confidence in the quality of data (Table I).

On the Fig.1 we plot the probability for simultaneous occurrence of more than one spanning cluster $P_L(k > 1) = \sum_{k>1} \pi_{h_k}$ versus the inverse system volume $1/L^2$. A linear fit gives us the limiting value of $P(k > 1) = 6.58 \cdot 10^{-3}$ with error $\approx 3 \cdot 10^{-5}$.

Fig.2 shows the dependence of the probability of more than two clusters versus $1/L^2$. The best linear fit gives the limiting value of $P(k > 2) = 1.48 \cdot 10^{-6}$ with uncertainty $2.1 \cdot 10^{-7}$.

Actually, we observed in computations mostly the events of up to three simultaneous spanning clusters. We simulated at total about 10^{10} samples of different sizes $8 \leq L \leq 64$ and only one sample with 4 spanning clusters was detected. This event clearly not contradicts with our estimate for $P(4) \approx 10^{-11}$, given below. Single event is probable as one part in ten.

We could fit exponent α in (2) taking $\alpha_k = -1/k^2 \log P(n \geq k)$ and obtaining $\alpha_1 = 0.693$, $\alpha_2 = 1.256$ and $\alpha_3 = 1.498$.

The logarithm of $P(k)$ is plotted on Fig.3 versus k^2 together with the best linear fit, which gives $\alpha \propto 1.61(7)$ and $A \propto 3 \pm 1.5$. For comparison, a linear fit for the logarithm of $P(k)$ versus k gives the value of exponent 2.8(5) with the too large uncertainty.

This is as full analysis as we could do because even with the using the of $\alpha = 1.5$ the probability of four ISC would be $3.8 \cdot 10^{-11}$ and of five ISC of $5.5 \cdot 10^{-17}$. Even the first value is impossible to check with the computers one

²This does not imply that we seriously expect any events with $k \geq 4$. The upper limit of eight is simply related to the numerical algorithm.

have today. To estimate accurately the probability of an event expected to be of the order 10^{-11} one needs to generate at least 10^{13} samples. Suppose, we have a computer with the CPU cycle of Δ ns and an algorithm which needs m CPU cycles to process one lattice site. Thus, it is $m \cdot L^2$ cycles per sample and $10^{13} \cdot m \cdot L^2$ cycles altogether. The time needed will be $10^{13} \cdot m \cdot L^2 \cdot \Delta$. If we are very optimistic, we could think that $m = 10$ and $\Delta = 1$ ns, and taking a moderate lattice size $L = 32$ one gets about 1157 ... days!

In addition, we compute the distribution of cluster sizes. In Fig.4 we plot the mean value $\langle s \rangle$ per lattice site of ISC of corresponding type. It is clear that the all clusters is of the same type, i.e. described by the same exponent, which is $\beta/\nu = 5/48 = 0.10416...$ as it should be [1] and as stressed in [9]. A linear fit gives the actual values 0.1056(14), 0.0999(20) and 0.1046(26), which are not far from the exact value. This is the additional argument that we are in force to summarize the probabilities π_{h_k} .

4 Discussion

We confirm more accurately the simulation result of [8, 9]³ for 2D percolation: the probability of two disjoint Incipient Spanning Clusters has a small but finite value. The difference in the values for probabilities is due to the fact that in [8, 9] helical boundary conditions are used. Whereas we used free boundary conditions.

The crossing probability with periodic boundary conditions (PBC) is known to be larger and the value 0.63665(8) is computed in [19]. We estimate in computer simulations the probability of disjoint ISC clusters to be $P(k > 1) \approx 2.0(4) \cdot 10^{-3}$ and $P(k > 2) \approx 1.4(5) \cdot 10^{-7}$ which are even smaller than for the case of free boundary conditions (FBC) considered by us. The finite size scaling of PBC is more complicated [19] and fit of the results for only four lattice sizes (8,16,32,64) we simulated gives less accurate limiting values in comparison with the one for FBC. Surprisingly, the linear fit of logarithm of $P(k)$ versus k^2 (shown in Fig. 4 by solid line) gives better accuracy then for the case of FBC: $\alpha_{PBC} \approx 1.915(1)$ and $A \approx 4.26(3)$.

It is necessary to note, that we assume the existence of the limiting probabilities (2) throughout this work and that the probabilities π_{h_k} with $k > 1$ reach their maximum in critical region. Indeed, our preliminary simulations [20] shows that the maximum of probabilities π_{h_1} and π_{h_2} occurs

³Actually, we got most of the results before getting the last preprint

at some p_{max} , which varies as $(p_{max} - p_c) \propto L^{-1/\nu}$ ($\nu = 4/3$ is the correlation length exponent) and that the limiting probabilities computed here seems to be correct.

Occasionally, from a total number of 10^{10} samples investigated by us (not all the data presented here), we detect the one event of sample with just four Incipient Spanning Clusters, which is as probable event as $1/10$.

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Table 1: Probabilities of more than k incipient spanning clusters on critical bond square lattices with linear size L and free boundaries. Note, that the values of $P(k > 1)$ is multiplyed by a factor 10^3 and those of $P(k > 2)$ by a factor 10^6 . For each lattice size L the first row is the probability of horizontal crossing and the second one is the probability of vertical crossing.

L	$k > 0$	$k > 1 \cdot 10^3$	$k > 2 \cdot 10^6$
8	0.50005(5)	7.657(8)	3.40(15)
	0.50003(4)	7.660(8)	3.98(21)
12	0.50002(5)	7.084(9)	2.57(14)
	0.49995(5)	7.070(8)	2.10(13)
16	0.50003(7)	6.855(9)	1.97(17)
	0.50002(6)	6.843(8)	1.79(19)
20	0.49990(6)	6.742(8)	1.95(14)
	0.50008(5)	6.745(9)	1.729(13)
30	0.49999(4)	6.650(8)	1.52(14)
	0.49996(5)	6.653(7)	1.52(12)
32	0.49999(5)	6.648(8)	1.73(12)
	0.50008(7)	6.642(8)	1.56(11)
64	0.49992(9)	6.597(9)	1.33(13)
	0.49999(6)	6.602(8)	1.51(14)
∞	0.50002(2)	6.58(3)	1.48(21)

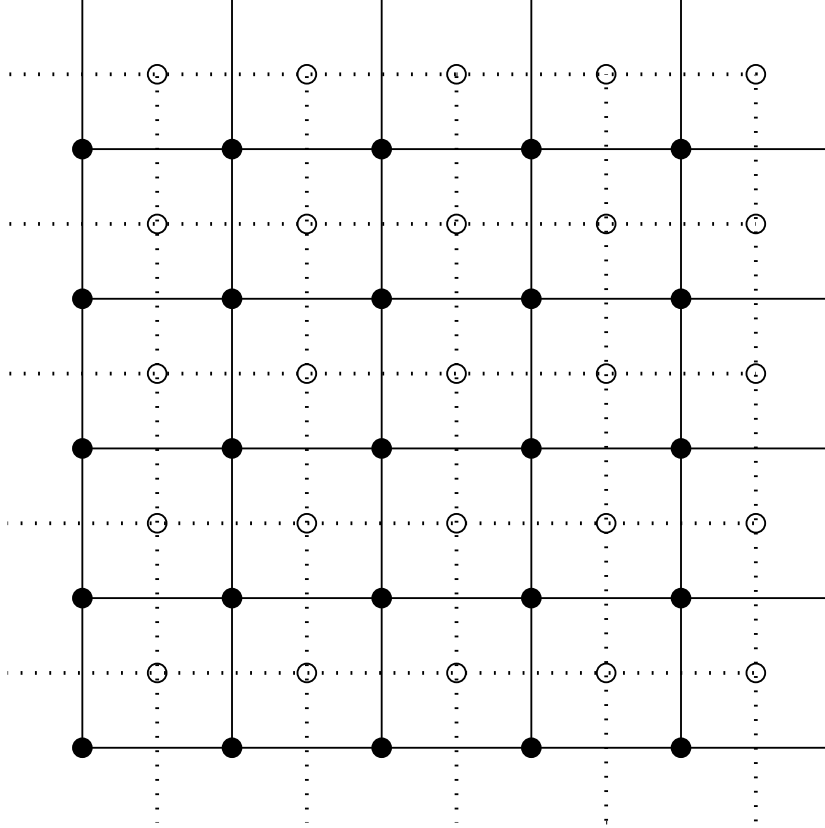


Figure 1: Example of simulated lattice with linear size $L = 5$ (solid lines) and its dual (dotted lines). Note, that the number of sites and bonds in both directions and for both lattices is just equal to L .

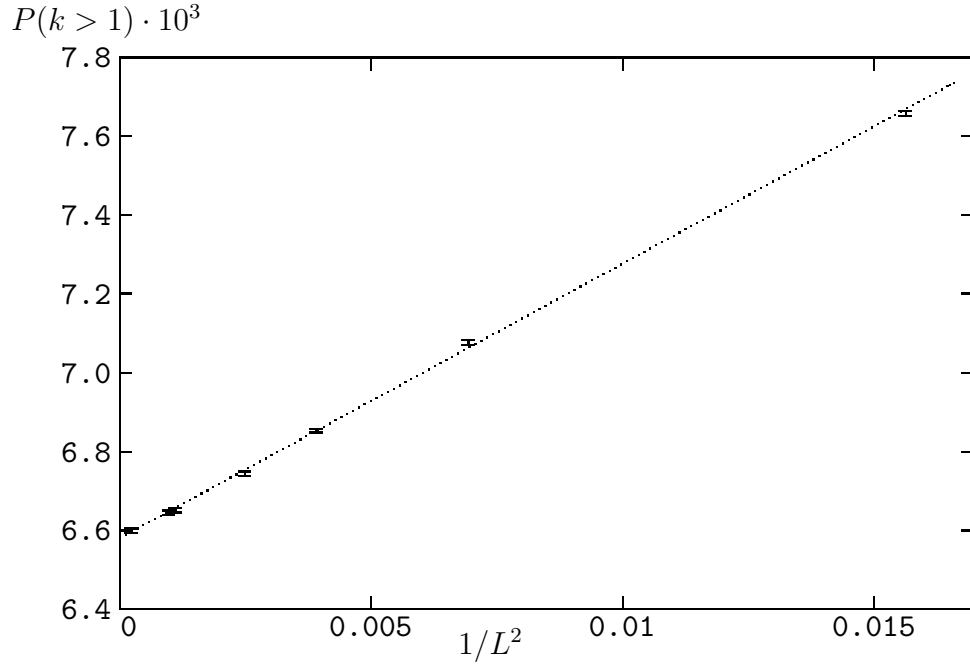


Figure 2: Probability of more than one Incipient Spanning Cluster multiplied by one thousand for 2D bond percolation model as a function of $1/L^2$. The linear lattice sizes L are 8,12,16,20,30,32,64. The probability approaches the value of 0.00658(53) in the limit of infinite L . Error bars are computed over 100 bins of 10^6 samples each.

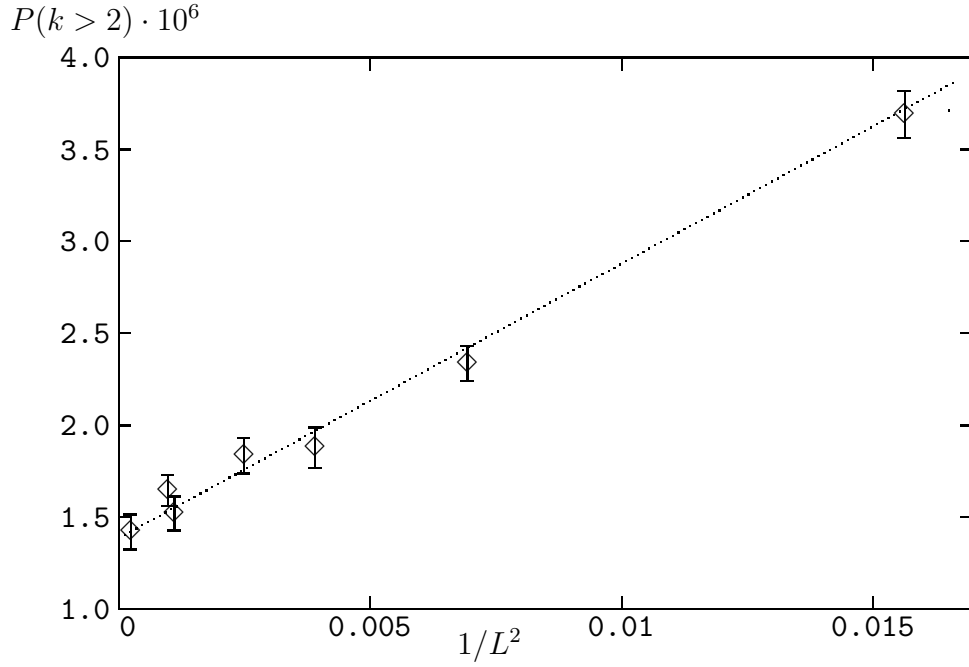


Figure 3: Probability of more than two Incipient Spanning Cluster multiplied by one million for 2D bond percolation model as a function of $1/L^2$. The linear lattice sizes L are 8,12,16,20,30,32,64. The probability approaches the value of 0.00000148(21) in the limit of infinite L . Error bars are computed over 100 bins of 10^6 samples each.

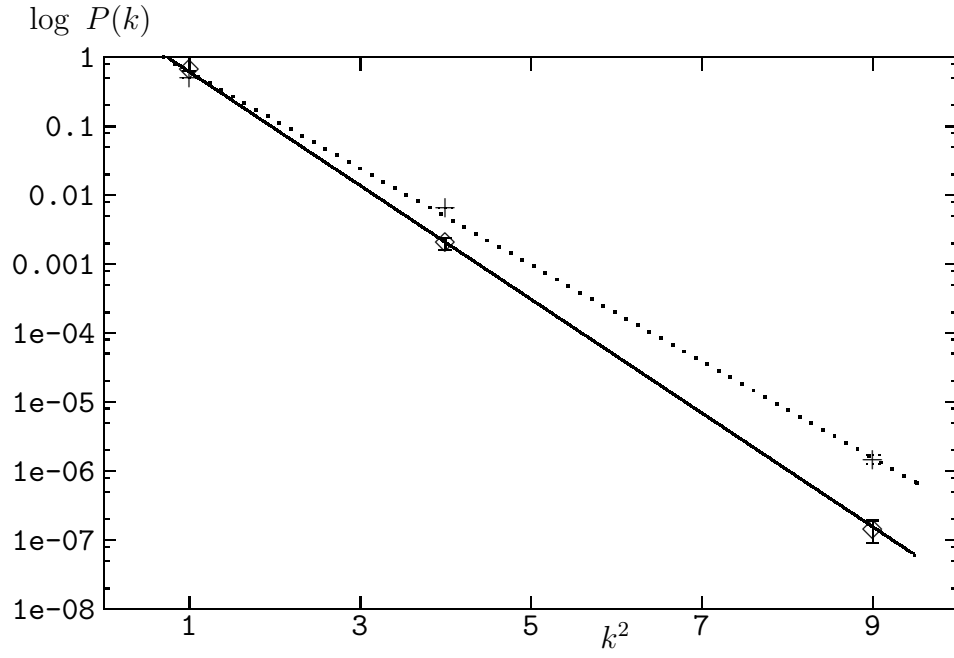


Figure 4: Estimated probabilities for more than k Incipient Spanning Clusters as function of k^2 in logarithmic scale. The dotted line is a fit for lattices with free boundaries and the solid line is a fit for the lattices with periodic boundaries in vertical direction.

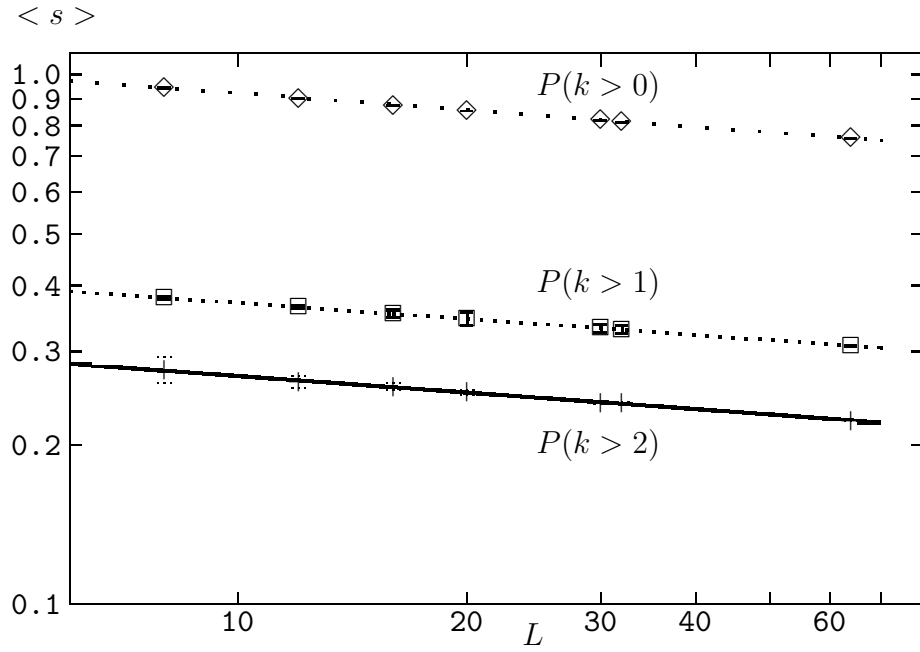


Figure 5: Mean size of k simultaneous Incipient Spanning Clusters, measured on the corresponding events. The slope of the curves is close to the exact value of β/ν for the all plotted curves.